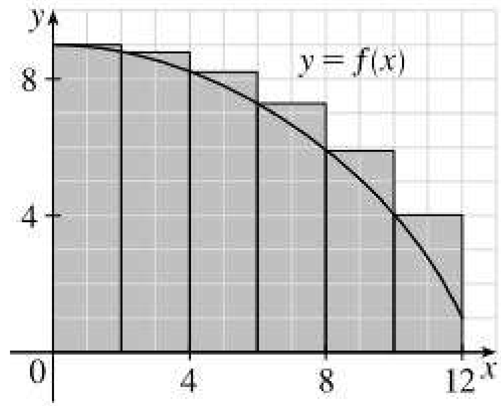
2. (a) (i) $L_6 = \sum_{i=1}^{6} f(x_{i-1})\,\Delta x \qquad [\Delta x = \frac{12-0}{6} = 2]$

$$= 2[f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)]$$

$$= 2[f(0) + f(2) + f(4) + f(6) + f(8) + f(10)]$$

$$\approx 2(9 + 8.8 + 8.2 + 7.3 + 5.9 + 4.1)$$

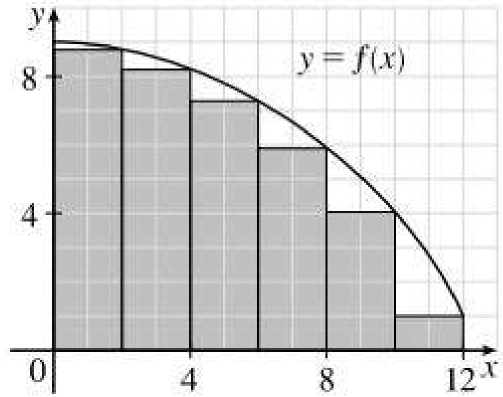
$$= 2(43.3) = 86.6$$



(ii) $R_6 = L_6 + 2 \cdot f(12) - 2 \cdot f(0)$

$$\approx 86.6 + 2(1) - 2(9) = 70.6$$

[Add area of rightmost lower rectangle and subtract area of leftmost upper rectangle.]

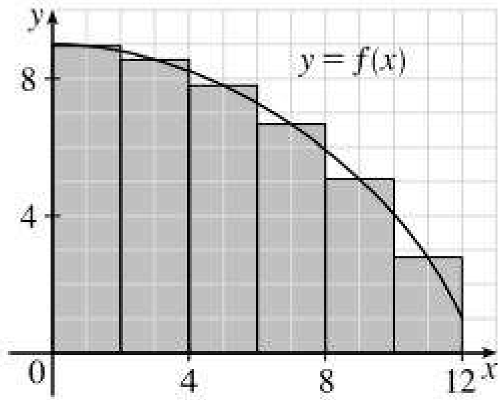


(iii) $M_6 = \sum_{i=1}^{6} f(\bar{x}_i)\,\Delta x$

$$= 2[f(1) + f(3) + f(5) + f(7) + f(9) + f(11)]$$

$$\approx 2(8.9 + 8.5 + 7.8 + 6.6 + 5.1 + 2.8)$$

$$= 2(39.7) = 79.4$$



(b) Since f is *decreasing*, we obtain an *overestimate* by using *left* endpoints; that is, L_6.

(c) Since f is *decreasing*, we obtain an *underestimate* by using *right* endpoints; that is, R_6.

(d) M_6 gives the best estimate, since the area of each rectangle appears to be closer to the true area than the overestimates and underestimates in L_6 and R_6.

1

General Parameters

- **Interval:** $[a, b] = [0, \pi]$
- **Width (Δx):** $\Delta x = \frac{\pi - 0}{n}$
- **Function:** $f(x) = 2 + \sin x$ is increasing on $[0, \pi/2]$ and decreasing on $[\pi/2, \pi]$.

1. For $n = 2$

- **Width:** $\Delta x = \pi/2$
- **Subintervals:** $[0, \pi/2]$ and $[\pi/2, \pi]$

Sum Type	Calculation	Result
Lower Sum (L_2)	$\Delta x[f(0) + f(\pi)]$	$\frac{\pi}{2}(2+2) = 2\pi \approx 6.28$
Upper Sum (U_2)	$\Delta x[f(\pi/2) + f(\pi/2)]$	$\frac{\pi}{2}(3+3) = 3\pi \approx 9.42$

2

2. For $n = 4$

- **Width:** $\Delta x = \pi/4$
- **Subintervals:** $[0, \pi/4], [\pi/4, \pi/2], [\pi/2, 3\pi/4], [3\pi/4, \pi]$
- **Lower Sum (L_4):** Uses the minimum value in each subinterval.

$$L_4 = \frac{\pi}{4}[f(0) + f(\pi/4) + f(3\pi/4) + f(\pi)]$$

$$L_4 = \frac{\pi}{4}[2 + (2 + \frac{\sqrt{2}}{2}) + (2 + \frac{\sqrt{2}}{2}) + 2] = \frac{\pi}{4}(8 + \sqrt{2}) \approx 7.39$$

- **Upper Sum (U_4):** Uses the maximum value in each subinterval.

$$U_4 = \frac{\pi}{4}[f(\pi/4) + f(\pi/2) + f(\pi/2) + f(3\pi/4)]$$

$$U_4 = \frac{\pi}{4}[(2 + \frac{\sqrt{2}}{2}) + 3 + 3 + (2 + \frac{\sqrt{2}}{2})] = \frac{\pi}{4}(10 + \sqrt{2}) \approx 8.96$$

3

3. For $n = 8$

As n increases, the gap between the upper and lower sums decreases, more closely approximating the true integral.

- **Width:** $\Delta x = \pi/8$
- **Lower Sum (L_8):** ≈ 7.84
- **Upper Sum (U_8):** ≈ 8.63

Summary Table

n	Lower Sum (L_n)	Upper Sum (U_n)
2	6.28	9.42
4	7.39	8.96
8	7.84	8.63

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The exact value of the integral is $2\pi + 2 \approx 8.28$. As $n \to \infty$, both sums will converge to this value according to **Theorem 4**.

(a) The velocities are given with units mi/h, so we must convert the 10-second intervals to hours:

$$10 \text{ seconds} = \frac{10 \text{ seconds}}{3600 \text{ seconds/h}} = \frac{1}{360} \text{ h}$$

$$\begin{aligned}\text{distance} \approx L_6 &= (182.9 \text{ mi/h})\left(\tfrac{1}{360} \text{ h}\right) + (168.0)\left(\tfrac{1}{360}\right) + (106.6)\left(\tfrac{1}{360}\right) + (99.8)\left(\tfrac{1}{360}\right) \\ &\quad + (124.5)\left(\tfrac{1}{360}\right) + (176.1)\left(\tfrac{1}{360}\right) \\ &= \frac{857.9}{360} \approx 2.383 \text{ miles}\end{aligned}$$

(b) Distance $\approx R_6 = \left(\tfrac{1}{360}\right)(168.0 + 106.6 + 99.8 + 124.5 + 176.1 + 175.6) = \dfrac{850.6}{360} \approx 2.363$ miles

(c) The velocity is neither increasing nor decreasing on the given interval, so the estimates in parts (a) and (b) are neither upper nor lower estimates.

1. Identify the Components

Based on the provided interval $[1, 3]$ and the formulas in Theorem 4:

- **Lower limit (a):** 1
- **Upper limit (b):** 3
- **Width of each subinterval (Δx):**

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$$

- **Right endpoint of the i-th subinterval (x_i):**

$$x_i = a + i\Delta x = 1 + i\left(\frac{2}{n}\right) = 1 + \frac{2i}{n}$$

2. Formulate the Expression

Definition 2 states that the area A is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

Substitute the function $f(x) = \frac{2x}{x^2+1}$ with the identified x_i and Δx:

- **Evaluate $f(x_i)$:**

$$f\left(1 + \frac{2i}{n}\right) = \frac{2\left(1+\frac{2i}{n}\right)}{\left(1+\frac{2i}{n}\right)^2 + 1}$$

- **Combine into the limit expression:**

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{2\left(1+\frac{2i}{n}\right)}{\left(1+\frac{2i}{n}\right)^2 + 1} \cdot \frac{2}{n}\right]$$

Final Expression

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4\left(1+\frac{2i}{n}\right)}{n\left[\left(1+\frac{2i}{n}\right)^2 + 1\right]}$$

$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n} \sqrt{1+\frac{3i}{n}}$ can be interpreted as the area of the region lying under the graph of $y = \sqrt{1+x}$ on the interval $[0, 3]$, since for $y = \sqrt{1+x}$ on $[0, 3]$ with $\Delta x = \frac{3-0}{n} = \frac{3}{n}$, $x_i = 0 + i\,\Delta x = \frac{3i}{n}$, and $x_i^* = x_i$, the expression for the area is $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+\frac{3i}{n}}\,\frac{3}{n}$. Note that this answer is not unique. We could use $y = \sqrt{x}$ on $[1, 4]$ or, in general, $y = \sqrt{x-n}$ on $[n+1, n+4]$, where n is any real number.